Mathematical Physics
Individual Contest

June 10, 2023

You can choose 3 out of the 4 questions.

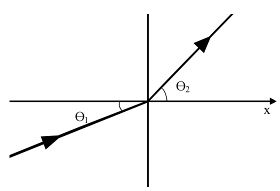
Question 1. Newtonian Mechanics

(1) In Newton's second law, if we set force $\vec{F} = 0$ we get acceleration $\vec{a} = 0$, which seems to give the Newton first law. Explain whether we need the first law as an independent postulate.

(2) A particle of mass M moves in the $x - y$ plane the piece-wise in presence of the constant potential

$$V = \begin{cases} V_2 & x > 0 \\ V_1 & x < 0 \end{cases}$$

In crossing from the domain $x < 0$, where its velocity $\vec{v}$ makes the angle θ_1 with the x-axis, to the domain $x > 0$, it changes its velocity (magnitude and angle).



What is the relation of θ_1 to θ_2 when $V_1 < V_2$ and $V_1 > V_2$?

Question 2. Consider a charged particle on a circle, parameterized by angle $\theta \sim \theta + 2\pi$. It moves under a constant magnetic field B perpendicular to the plane of the circle. The Lagrangian of the system is given by

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{B}{2\pi}\dot{\theta}\,.$$

(1) Find the spectra and corresponding wave functions of the system (set $\hbar = 1$), and show the ground states have a two-fold degeneracy for $B = \pi$.

(2) For $B = \pi$, the system admits a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry generated by $\theta \to \theta + \pi$ and $\theta \to -\theta$. Find how they act on the two-fold ground states and the commutation relation of the two operators, and thus show that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is central extended to the dihedral group with 8 elements

$$\mathbb{D}_8 = \left\langle r, s \,|\, r^4 = s^2 = 1, s\, r\, s = r^3 \right\rangle .$$

(3) Suppose now the particle moves under an extra potential

$$V = \lambda \cos(2\,\theta).$$

Show that there are still two ground states perturbatively. Use the above results to argue that the two-fold ground states cannot be lifted non-perturbatively for generic λ, and explicitly verify it via an one-instanton computation (i.e. the leading contribution as $\hbar \to 0$ to the tunnelling amplitude between the two perturbative vacua).

Question 3. We introduce the spinor-helicity formalism for massless particles. We write any four-momentum p^μ as a 2×2 matrix $\mathbf{p}^{\dot\alpha\alpha}$ via

$$p^\mu \to \mathbf{p}^{\dot\alpha\alpha} = \bar\sigma^{\dot\alpha\alpha}_\mu p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \quad \bar\sigma^{\dot\alpha\alpha}_\mu = (1, \vec\sigma), \tag{1}$$

where $\vec\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the three 2×2 Pauli matrices. For massless particles, the matrix $\mathbf{p_i}^{\dot\alpha\alpha}$ can be written as a product of two 2-component vectors $\tilde\lambda_i^{\dot\alpha}$ and λ_i^α which are called the spinor helicity variables, *i.e.*

$$\mathbf{p_i}^{\dot\alpha\alpha} = \tilde\lambda_i^{\dot\alpha}\lambda_i^\alpha. \tag{2}$$

Any Lorentz-invariant can be built from these λ_i and $\tilde\lambda_i$ through the angle $\langle..\rangle$ and square $[..]$ brackets defined as

$$\begin{aligned} \langle\lambda_i\lambda_j\rangle &:= \varepsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta = -\langle\lambda_j\lambda_i\rangle =: \langle ij\rangle, \\ \left[\tilde\lambda_i\tilde\lambda_j\right] &:= -\varepsilon_{\dot\alpha\dot\beta}\tilde\lambda_i^{\dot\alpha}\tilde\lambda_j^{\dot\beta} = -[\lambda_j\lambda_i] =: [ij], \end{aligned} \tag{3}$$

where the $\varepsilon_{\alpha\beta}$ is the rank 2 Levi-Civita tensor.

We also define the helicity operator for the n-point amplitude as

$$\hat h = \frac{1}{2}\sum_{i=1}^n \left(-\lambda_i^\alpha \frac{\partial}{\partial\lambda_i^\alpha} + \tilde\lambda_i^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}} \right) \equiv \sum_{i=1}^n \hat h_i, \tag{4}$$

which means we have assigned helicity $-\frac{1}{2}$ to λ and helicity $\frac{1}{2}$ to $\tilde\lambda$.

(1) Show the on-shell condition can be expressed as

$$p_i^2 = m^2 \quad \Leftrightarrow \quad \det\left(\mathbf{p_i}^{\dot\alpha\alpha}\right) = m^2. \tag{5}$$

In particular, for a massless particle, prove that $\mathbf{p_i}^{\dot\alpha\alpha}$ can be written as in Equation (2).

(2) Show that for two general four-vectors A^μ and B^μ, their Lorentz product $A \cdot B$ can be written in terms of the matrix $\mathbf{A}^{\dot{\alpha}\alpha}, \mathbf{B}^{\dot{\alpha}\alpha}$; write the so-called Mandelstam variable $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$ for two massless particles in terms of square and angle brackets.

(3) Consider the 3-point amplitude where the helicity of particle i is h_i. Suppose it only depends on angle brackets, we can then write an ansatz

$$A_3\left(1^{h_1} 2^{h_2} 3^{h_3}\right) = c\langle 12\rangle^{x_{12}} \langle 13\rangle^{x_{13}} \langle 23\rangle^{x_{23}}, \tag{6}$$

where c is a constant independent of the kinematics. Express x_{12}, x_{13}, x_{23} in terms of h_i, and write down three-gluon amplitude $A_3\left(1^- 2^+ 3^-\right)$.

Question 4. Consider the 3d pure gravity theory with the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N} \int_M \mathrm{d}x^3 \sqrt{-g}\, R\,.$$

Recast the action in terms of vierbein e^a_μ and spin-connection $\omega_{\mu\,b}^{\,a}$, and further show its equivalence to a Chern-Simon theory with action

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A\mathrm{d}A + \frac{2}{3}A^3\right)\,,$$

where the gauge field A takes values in the Lie algebra of $\mathrm{ISO}(1,2)$.